MATH2050C Selected Solution to Assignment 12

Section 5.4 no. 3, 4, 6, 7, 8, 10, 15. **Section 5.6** no 3,4.

Section 5.4

(3) (a) $f(x) = x^2$ is not uniformly continuous on $[0,\infty)$. Pick $a_n = n$ and $b_n = n + 1/n$. Then $|a_n - b_n| = 1/n \to 0$ but $|f(a_n) - f(b_n)| = 2 + 1/n^2 > 2$.

Note. In general, any polynomial of degree ≥ 2 is not uniformly on any unbounded interval. (Of course, it is uc on every bounded interval.)

(b) $g(x) = \sin 1/x$ on $(0,\infty)$. Pick $a_n = 1/(2n\pi)$ and $b_n = 1/(2n+1/2)\pi$. Then $|a_n - b_n| \to 0$ but $|\sin 1/a_n - \sin 1/b_n| = |0-1| = 1$ for all n.

(4) Observing f is decreasing, on an interval of the form $I = [x, x+\delta], x \geq 0$, its oscillation is given by

$$\mathrm{osc}_I f = \frac{1}{1+x^2} - \frac{1}{1+(x+\delta)^2} = \frac{2\delta x + \delta^2}{(1+x^2)(1+(x+\delta)^2)} \ .$$

For $\delta \leq 1$,

$$\frac{2\delta x + \delta^2}{(1+x^2)(1+(x+\delta)^2)} \leq \frac{2\delta x + \delta^2}{1+x^2} \leq 2\delta \ ,$$

as $2x \leq 1 + x^2$ and $\delta^2 \leq h$. Hence given $\varepsilon > 0$, pick $\delta = \min\{1, \varepsilon/2\}$, we have $\mathrm{osc}_f \leq 2\delta \leq \varepsilon$, on $[x, x+\delta], x \geq 0$. By the Oscillation Theorem f is uniformly continuous on $[0,\infty)$.

(6) Let f be bounded by M and g by K. Use

$$|f(x)g(x) - f(y)g(y)| = |(f(x)-f(y))g(x) + f(y)(g(x)-g(y))| \leq K|f(x)-f(y)| + M|g(x)-g(y)| \ .$$

(7) The functions x and $\sin x$ are uniformly continuous on $(-\infty,\infty)$, but its product $h(x) = x\sin x$ is not. Let $a_n = 2n\pi$ and $b_n = (2n+1/n)\pi$ so $|a_n - b_n| \to 0$. On the other hand,

$$\frac{\sin\left(2n\pi + \dfrac{1}{n}\pi\right)}{\pi/n} = \frac{\sin \dfrac{\pi}{n}}{\pi/n} \to 1 \ , \quad \text{as } n \to \infty \ .$$

Thus,

$$|b_n \sin b_n - a_n \sin a_n| = |b_n \sin b_n| \to 2\pi^2 \ , \qquad \text{as } n \to \infty.$$

(8) Same as the proof of the composite of two continuous functions is continuous, just noting that δ depends on ε only.

(10) If not, there is a sequence $\{x_n\}$ in A such that $|f(x_n)| \geq n$. As A is bounded, by Bolzano-Weierstrass, by passing to a subsequence if nec, we may assume $x_n \to x^*$ for some x^* (not nec in A). Then $\{x_n\}$ is a Cauchy sequence. Now, by assumption f is uniformly continuous, for $\varepsilon = 1$, there is some δ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. As $\{x_n\}$ is a Cauchy sequence, $|x_n - x_m| < \delta$ for all $n, m \geq n_0$. But then

$$n \leq |f(x_n)| \leq |f(x_n) - f(x_{n_0})| + |f(x_{n_0})| \leq 1 + |f(x_{n_0})| \ ,$$

which is impossible for large n. Hence, f must be bounded.

(15) (c) An example is the linear function $f(x) = x$. Clearly it is Lipschitz continuous, but x^2 is not.

Section 5.6

(3) It is clear that both functions are strictly increasing everywhere. Their product $h(x) = x(x-1)$ satisfies $h(0) = h(1) = 0$ so it cannot be increasing on $[0, 1]$. Indeed, if h is increasing, it implies that h is the constant zero function which is clearly ridiculous. In general, the product of two non-negative, increasing functions is increasing.

(4) Let f and g be two positive, increasing function and let $x < y$ be two points in their domain of definition. Then,

$$(fg)(x) - (fg)(y) = f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)) \leq 0 \ ,$$

so fg is increasing.

Supplementary Problems

1. Let function f on E satisfy the condition: There is some constant C and $\alpha > 0$ such that $|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$ for all $x \in E$. (It is called Lipschitz continuous when $\alpha = 1$.) Show that f is uniformly continuous on E.

   Solution Given $\varepsilon > 0$, take $\delta < (\varepsilon/C)^{1/\alpha}$ to get the job done.

2. Let f be a uniformly continuous function on $[0, \infty)$. Show that there is a constant C such that $|f(x)| \leq C_1 + C_2 x$.

   Solution For $\varepsilon = 1$, there is some $\delta > 0$ such that $|f(x) - f(y)| < 1$ for $x, y, |x - y| \leq \delta$. Decompose $[0, \infty)$ into $[(n-1)\delta, n\delta], n \geq 1$. Let $C_0 = \sup_{x \in [0,1]} |f(x)|$. Then $|f(2\delta)| \leq |f(2\delta) - f(\delta)| + |f(\delta)| \leq 1 + C_0$. By induction we have $|f(n\delta)| \leq C_0 + n$ for all n. Now, given $x > 0$, there is some n such that $(n-1)\delta \leq x < n\delta$, hence

   $$|f(x)| \leq |f((n-1)\delta)| + 1 \leq C_0 + n - 1 + 1 \leq C_0 + \frac{x}{\delta} + 1 \leq C_1 + C_2 x \ ,$$

   where $C_1 = C_0 + 1$ and $C_2 = 1/\delta$.

3. (Optional) Order the rational numbers in $(0, 1)$ into a sequence $\{x_k\}$. Define a function on $(0, 1)$ by $\varphi(x) = \sum 1/2^k$ where the summation is over all indices k such that $x_k < x$. Show that

   (a) φ is strictly increasing and $\lim_{x \to 1^-} \varphi(x) = 1$.

   (b) φ is discontinuous at each x_k.

   (c) φ is continuous at each irrational number in $(0, 1)$.

Solution A sketchy proof. (a) It is obvious that φ is strictly increasing and $\lim_{x\to 1^-}\varphi(x) = 1$ since $\sum_{k=1}^{\infty} 2^{-k} = 1$.

(b) Observe that $j_\varphi(x_k) \geq 2^{-k} > 0$.

(c) Given $\varepsilon > 0$, fix a large k_0 such that $\sum_{k=k_0+1}^{\infty} 2^{-k} < \varepsilon$. Let $z \in (0,1)$ be irrational. We can find a small δ such that $(z-\delta, z+\delta)$ does not contain any x_k with index $k \leq k_0$. Then for $x < y, x, y \in (z-\delta, z+\delta)$,

$$0 < \varphi(y) - \varphi(x) \leq \sum_{k=k_0+1} 2^{-k} < \varepsilon \ ,$$

hence φ is continuous at z.

Note This example shows how complicated a monotone function could be.